Episode 8

Linear Momentum

ENGN0040: Dynamics and Vibrations
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Topics for todays class

Impulse-Momentum relations for systems of particles

1. Definitions of linear impulse and linear momentum
2. Impulse-Momentum relations for a single particle
3. Calculating momentum of a system of particles
4. Impulse-Momentum relations for a system of particles
5. Applications

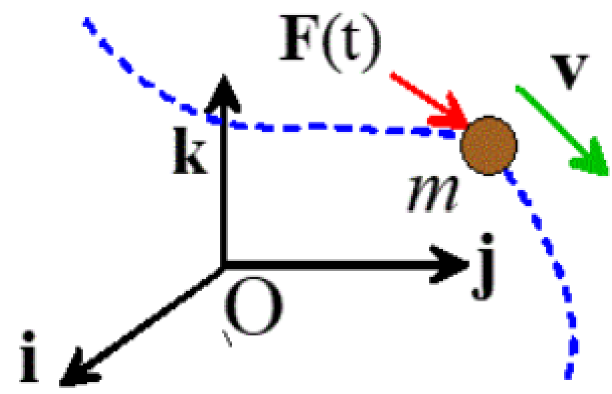
4·3 Impulse – Momentum relations for a single particle

4·3·1 Definitions of linear impulse and momentum

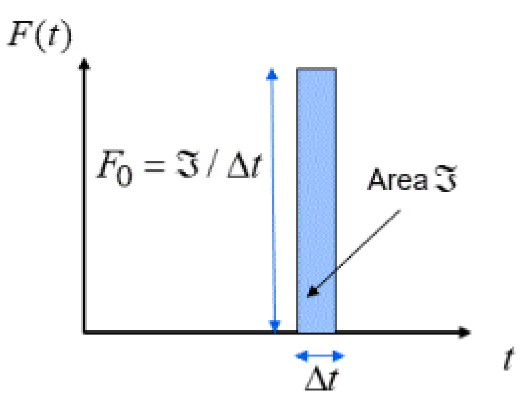
Impulse of a force

$$\underline{J} = \int_{t_0}^{t_1} \underline{F}(t)\, dt$$

Units N-s



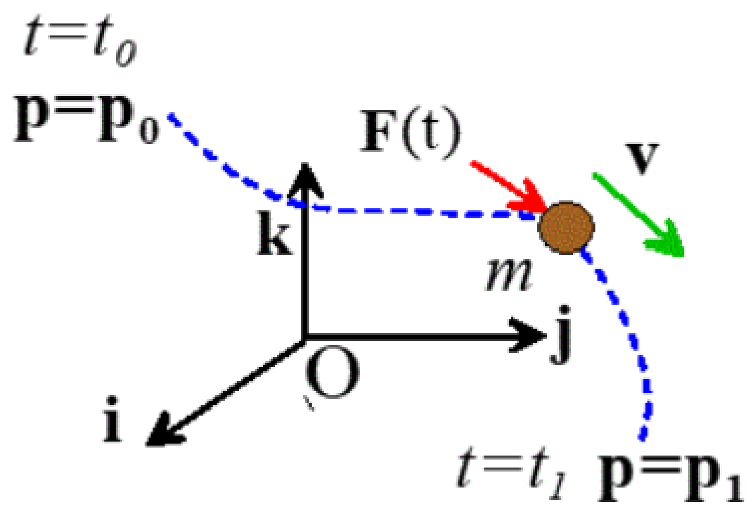
"An Impulse": Very large force applied for very short time interval. Eg hammer blow



$\underline{J}$: impulse vector, units Ns

Linear momentum of a particle $\underline{p} = m\underline{v}$

4.3.2 Impulse-momentum relation for a single particle



Version 1

$$\boxed{\underline{F} = \frac{d\underline{p}}{dt}}$$

Version 2

$$\boxed{\underline{J} = \underline{p}_1 - \underline{p}_0}$$

Proof

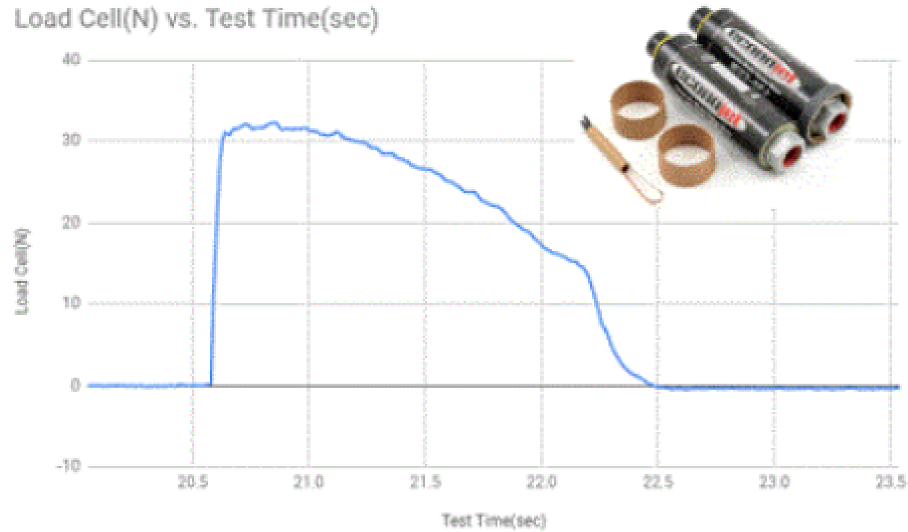
$$\underline{F} = m\underline{a} = m\frac{d\underline{v}}{dt} = \frac{d}{dt}(m\underline{v}) = \frac{d\underline{p}}{dt}$$

Separate variables and integrate

$$\int_{t_0}^{t_1} \underline{F}\, dt = \int_{p_0}^{p_1} d\underline{p}$$

$$\Rightarrow \quad \underline{J} = \underline{p}_1 - \underline{p}_0$$

4.3.3: Example: Calculate the impulse exerted by a model rocket motor from data provided in a csv file

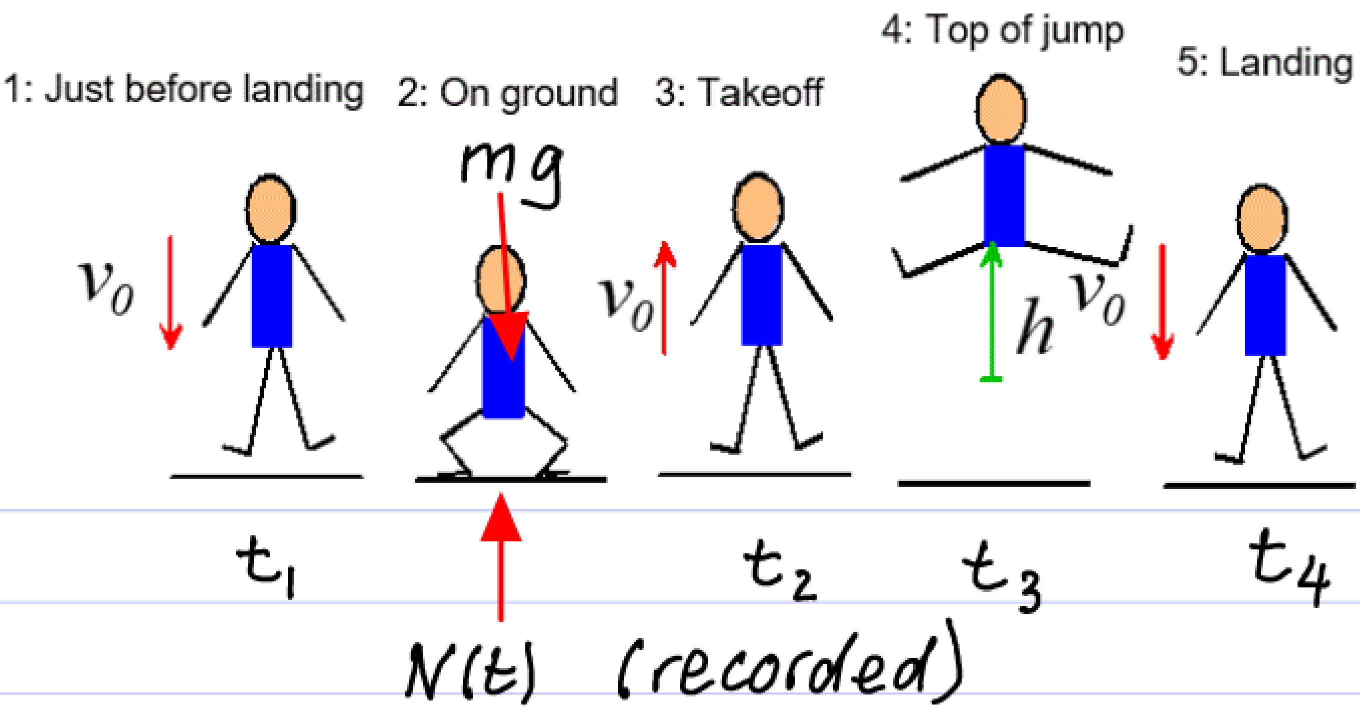


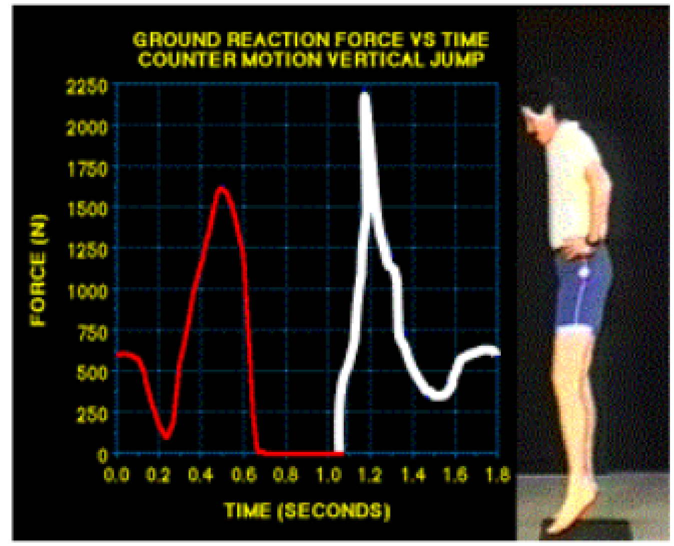
Definition $$J = \int_{t_0}^{t_1} F \, dt$$

Read file with MATLAB & integrate numerically

Solution $$\boxed{J = 42.6 \text{ Ns}}$$

4.3.3: Example: Estimate height of a jump from a force-plate measurement of reaction forces





Approach:

(1) Assume steady state jumping

(2) Energy conservation ⇒ same speed just before & just after jump

(3) Use impulse-momentum to get v_0

(4) Use energy to get h

① Impulse-momentum $\underline{J} = \underline{p}_1 - \underline{p}_0 \qquad t_1 < t < t_2$

Here

$$\underline{J} = \int_{t_1}^{t_2} [N(t) - mg]\,\underline{j}\,dt$$

$$= \left\{ \int_{t_1}^{t_2} N(t)\,dt - mg(t_2 - t_1) \right\} \underline{j}$$

$$\underline{p}_1 = m v_0 \underline{j} \qquad \underline{p}_0 = m(-v_0 \underline{j})$$

Use formula

$$\left\{ \int_{t_1}^{t_2} N(t)\,dt - mg(t_2 - t_1) \right\} \underline{j} = m v_0 \underline{j} - m(-v_0 \underline{j})$$

$$\Rightarrow \boxed{V_0 = \frac{1}{2m} \int_{t_1}^{t_2} N(t)\,dt - g\frac{(t_2 - t_1)}{2}}$$

(2) Energy method to find h

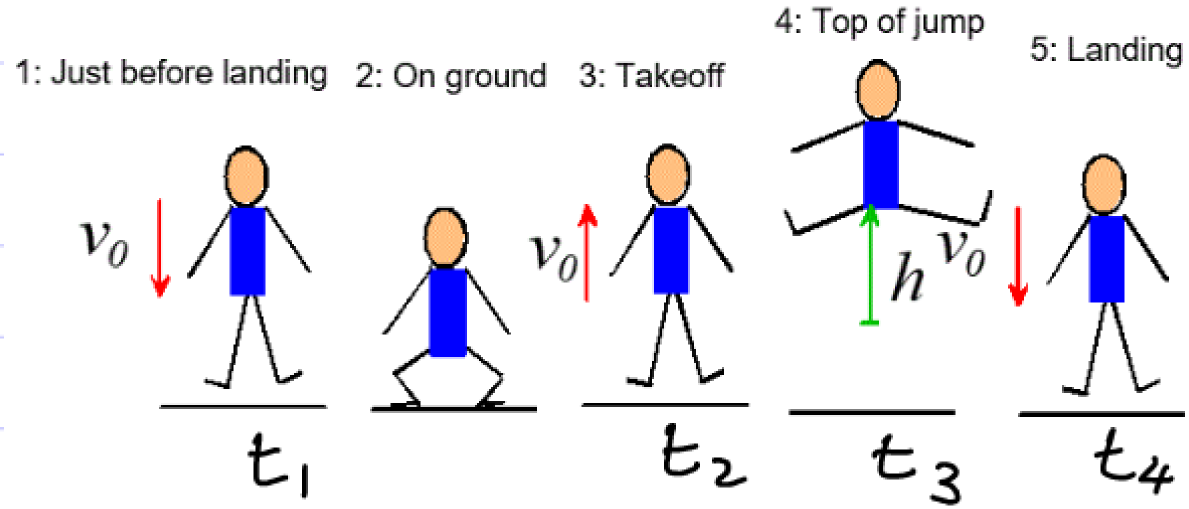
System : earth + jumper
Conservative ; no ext forces
State (0) @ time t_2
State (1) @ time t_3

$$T_1 + U_1 = T_0 + U_0$$

$$\Rightarrow \quad 0 + mgh = \frac{1}{2} m v_0^2 + 0$$

$$\Rightarrow \quad \boxed{h = \frac{v_0^2}{2g}}$$

Check Use measured time in air to calculate V_0



By symmetry

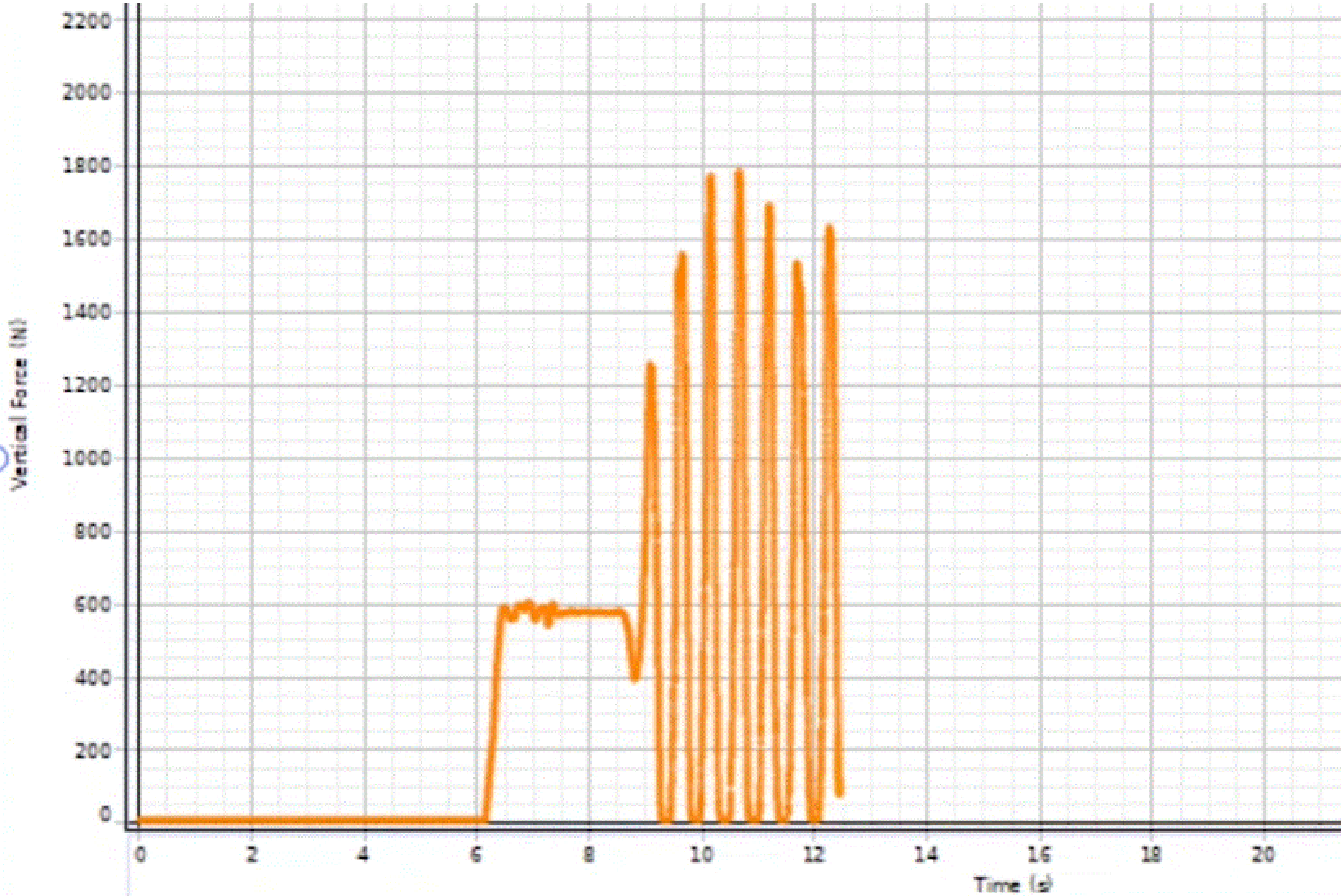
$$t_3 - t_2 = t_4 - t_3$$

Straight line motion formula for $t_2 < t < t_3$

$$0 = V_0 - g(t_3 - t_2)$$

$$\Rightarrow \boxed{V_0 = g\,\frac{(t_4 - t_2)}{2}}$$

Experimental Data



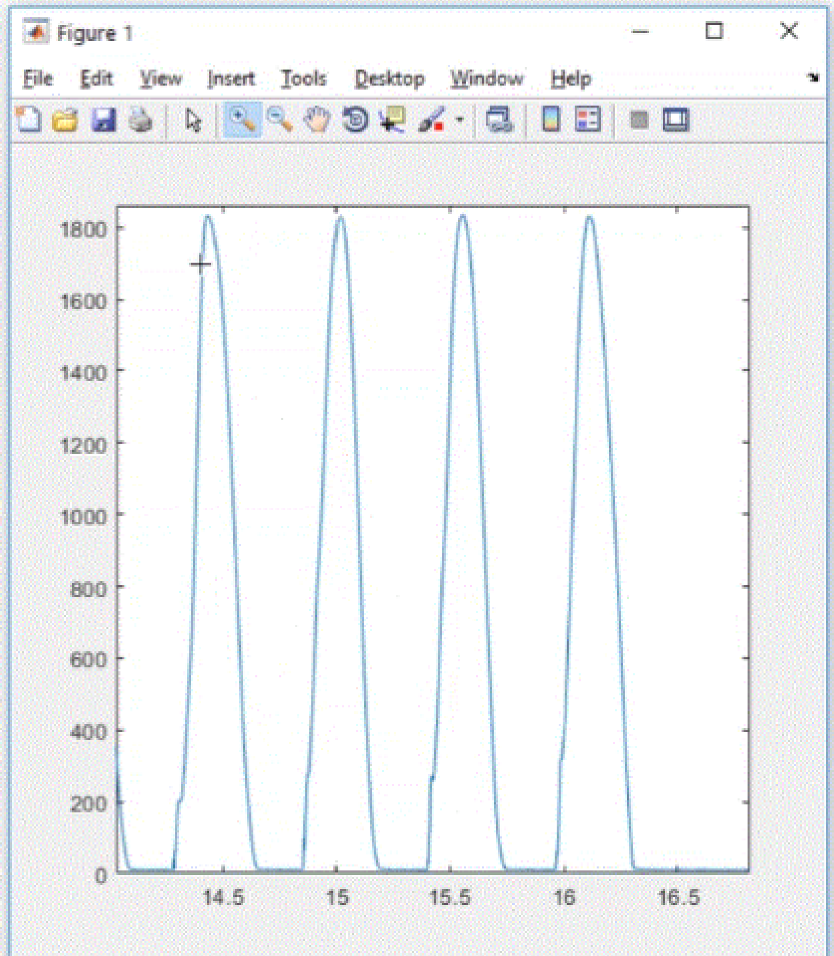
Process data with MATLAB

```
function process_jump
  close all
  g = 9.81; m = 579.5/g;
  data = csvread('Forceplate_Data.csv',1,0);
  time = data(:,1); force = data(:,2);
  plot(time,force);
  mask = time>14.75 & time<15.3;
  t_c = time(mask); f_c = force(mask)-10;
  plot(t_c,f_c)
  mask2 = time>15 & time<15.5;
  t_a = time(mask2); f_a = force(mask2)-10;
  plot(t_a,f_a)

  impulse = trapz(t_c,f_c);
  mask3 = f_c>5;
  t_onground = t_c(mask3);
  t1 = t_onground(1); t2 = t_onground(end);
  mask4 = f_a<5;
  t_inair = t_a(mask4);
  t4 = t_inair(end);
  v0_f = impulse/(2*m) - g*(t2-t1)/2;
  v0_a = g*(t4-t2)/2;
  h_f = v0_f^2/(2*g)
  h_a = v0_a^2/(2*g)

end
```

From force-plate $\boxed{V_0 = \frac{1}{2m}\int_{t_1}^{t_2} N(t)\,dt - g\,\frac{(t_2-t_1)}{2}}$

From time in air $\boxed{V_0 = g\,(t_4-t_2)/2}$

$\boxed{h = V_0^2/2g}$



Forceplate	Time in air
h_f = 0.0578 m	h_a = 0.0567 m

4.4 Impulse-Momentum relations for a system of particles

Preliminaries: Describing forces & impulse on system

Note: Newton =>

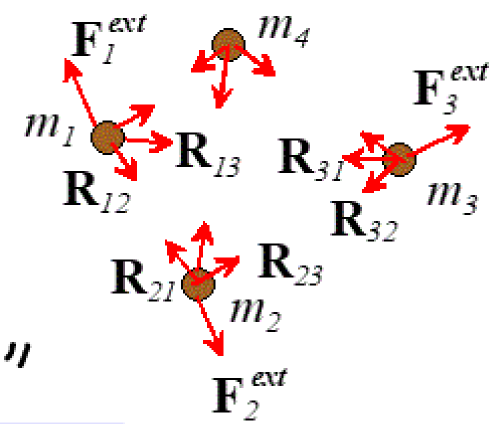
$\underline{R}_{12} = -\underline{R}_{21}$

$\underline{R}_{13} = -\underline{R}_{31}$

etc

$\underline{R}_{ij} = -\underline{R}_{ji}$

$\mathbf{R}_{ij}$	Force exerted on particle i by particle j
$\mathbf{F}_i^{ext}$	External force on particle i
$\mathbf{v}_i$	Velocity of particle i



$\underline{R}_{ij}$: "Internal Forces"

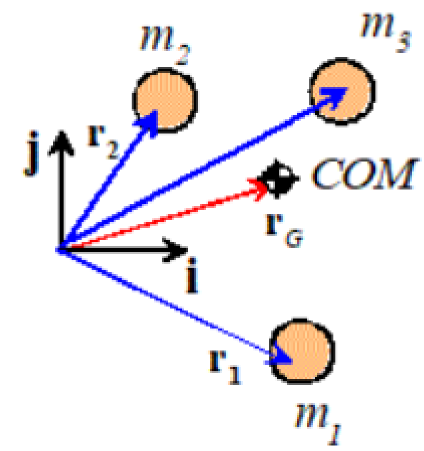
$\underline{F}_i^{ext}$: "External Forces"

Total External Impulse $\underline{J}^{ext} = \int_{t_0}^{t_1} \sum_{\substack{ext \\ forces}} \underline{F}_i^{ext}(t)\, dt$

4.4.1 Total momentum of a system

Direct summation $\boxed{\underline{p}^{TOT} = \sum_{particles} m_i \underline{v}_i}$

Using center of mass



Define Total mass $M = \sum_{particles} m_i$

Center of mass $\underline{r}_G = \frac{1}{M} \sum_{particles} m_i \underline{r}_i$

Let $\underline{v}_G = \frac{d\underline{r}_G}{dt}$ (velocity of COM)

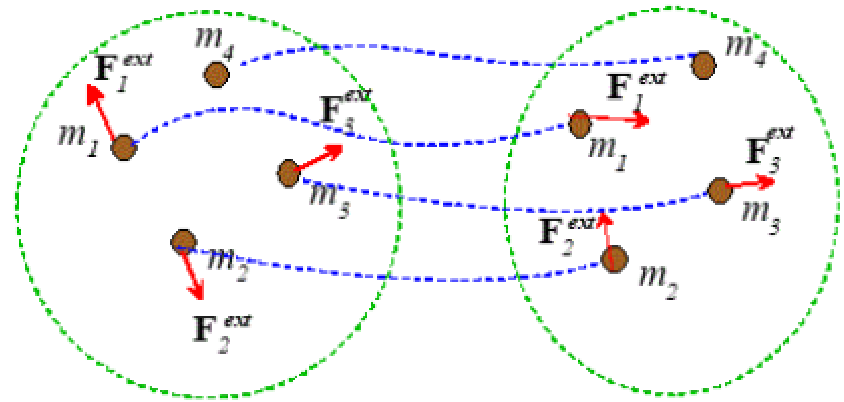
Then $\boxed{\underline{p}^{TOT} = M \frac{d\underline{r}_G}{dt} = M \underline{v}_G}$

Proof $\underline{p}^{TOT} = \sum m_i \underline{v}_i = \frac{d}{dt} \sum m_i \underline{r}_i = M \frac{d\underline{r}_G}{dt}$

4.4.2 Impulse - Momentum relations for a system

Total External Force $\mathbf{F}^{TOT}(t)$

Total External Impulse $\Im^{TOT} = \int_{t0}^{t1} \mathbf{F}^{TOT}(t)dt$



$t = t_0$ *Total momentum* $\mathbf{p}_0^{TOT}$

$t = t_1$ *Total momentum* $\mathbf{p}_1^{TOT}$

Version 1

$$\sum_{\substack{ext\\forces}} \underline{F}_i^{ext} = \frac{d\underline{p}^{TOT}}{dt}$$

Version 2

$$\underline{J}^{ext} = \underline{p}_1^{TOT} - \underline{p}_0^{TOT}$$

Special Case: $\underline{J}^{ext} = \underline{0}$

$$\underline{p}_1^{TOT} = \underline{p}_0^{TOT}$$

Total momentum is conserved

Proof

① Impulse - Momentum for one particle

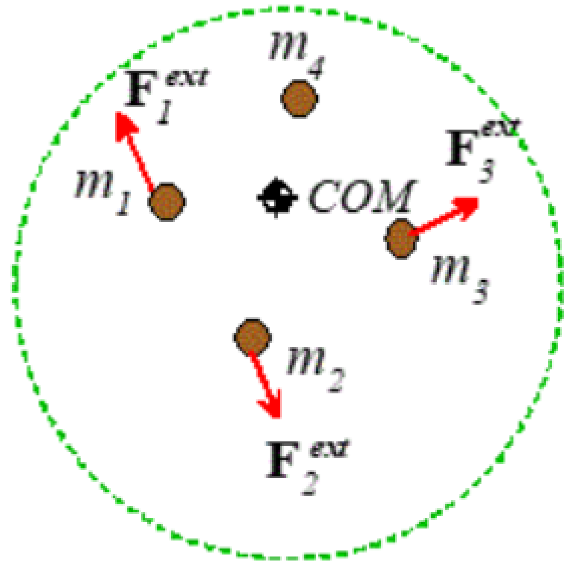
$$\underset{\sim}{F}_i^{ext} + \sum_{\text{particles } j} \underset{\sim}{R}_{ij} = \frac{d\underset{\sim}{p}_i}{dt}$$

② Sum over all particles

$$\underbrace{\sum_{\text{particles}} \underset{\sim}{F}_i}_{\text{Total ext force}} + \underbrace{\sum_i \sum_j \underset{\sim}{R}_{ij}}_{\substack{\text{Sum of all } \underset{\sim}{R}_{ij} = \underset{\sim}{0} \\ \text{since } \underset{\sim}{R}_{ij} = -\underset{\sim}{R}_{ji}}} = \underbrace{\frac{d}{dt} \sum \underset{\sim}{p}_i}_{\frac{d\underset{\sim}{p}^{TOT}}{dt}}$$

Proves version ①. For ②, just integrate ①

Useful Observation



Center of mass of system obeys Newtons Law

Recall: (1) $\underline{p}^{TOT} = M \underline{v}_G$

(2) $\sum\limits_{\substack{\text{ext} \\ \text{forces}}} \underline{F}_i^{ext} = \frac{d\underline{p}^{TOT}}{dt} = M \frac{d\underline{v}_G}{dt} = M \underline{a}_G$

If no external forces act COM moves at constant velocity (or is often stationary)

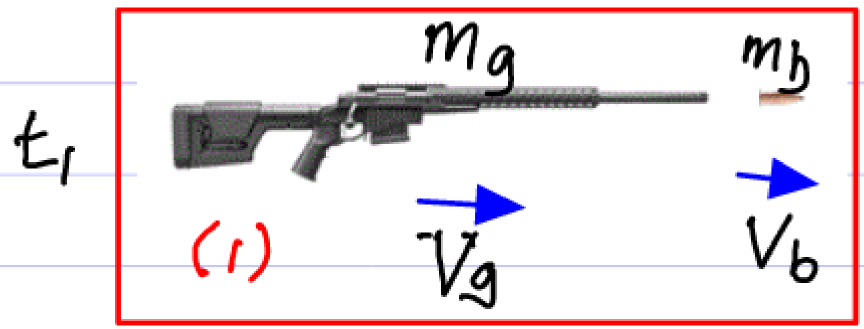
4.4.3: **Example:** Estimate the recoil velocity of a rifle

Remington Model 700 PCR

Data:

Rifle mass 4.8kg
6.5mm Creedmoor cartridge, 140 grain (9.1 gram) bullet
820 m/s muzzle velocity

System = gun + bullet
State (0) : just before firing
State (1) : just after firing



$$\underline{J}^{ext} = \int_{t_0}^{t_1} \underline{F}^{ext} \, dt$$

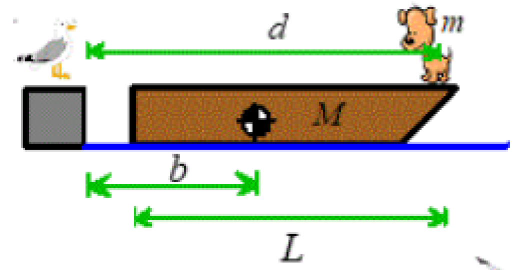
$\underline{F}^{ext}$ is finite and $t_1 - t_0 \ll 1 \Rightarrow \underline{J}^{ext} \approx \underline{0}$

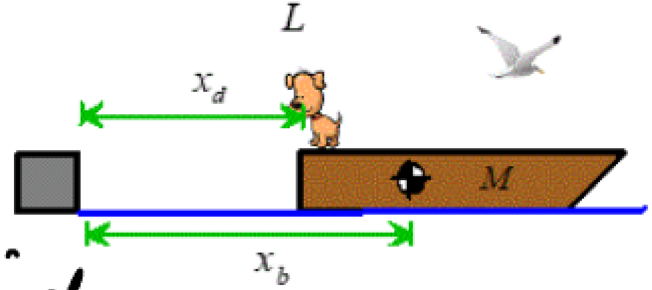
Hence $\underline{p}_1^{TOT} = \underline{p}_0^{TOT}$

$$m_g V_g \underline{i} + m_b V_b \underline{i} = \underline{0} \Rightarrow V_g = -\frac{m_b}{m_g} V_b$$

$$\Rightarrow \boxed{V_g \sim 1.5 \text{ m/s}} \quad (\sim 5 \text{ ft/s})$$

4.4.4: Example: A dog with mass *m* sits in the bow of a boat with length *L* and mass *M*. The dog is a distance *d* from the dock. It then walks to the stern of the boat. How far is the dog from the dock?





System = boat + dog

Boat floats freely ⇒ no horizontal ext force ⇒ horizontal pos of COM fixed

$$x_{com} = \frac{1}{(M+m)}(Mb + md) = \frac{1}{(M+m)}(Mx_b + mx_d) \quad (1)$$

Geometry

$$(x_b - b) + (d - x_d) = L \quad (2)$$

$$(1) \Rightarrow \quad -M(x_b - b) + m(d - x_d) = 0 \quad (3)$$

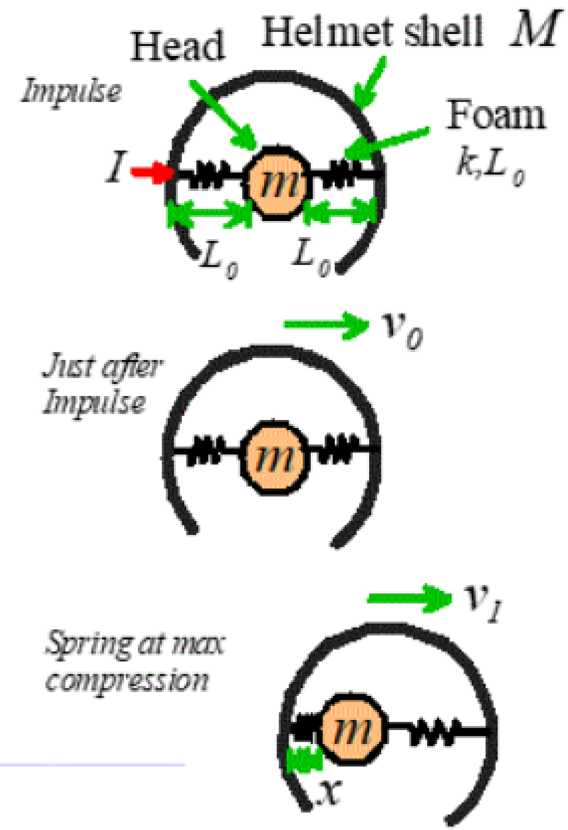
$$M(2) + (3) \Rightarrow (M+m)(d - x_d) = ML$$

$$\Rightarrow \quad \boxed{x_d = d - \frac{ML}{(M+m)}}$$

4.4.5: Example: A helmet shell with mass M is padded with foam with stiffness k and thickness L_0. A head with mass m is inside the helmet. The system is at rest for $t<0$

At time t=0 the helmet shell is struck with an impulse I. Find formulas for:

(a) The speed of the helmet shell just after the impact
(b) The length of the padding x at the instant of maximum compression
(c) The maximum acceleration of the head $a_{\max}$
(d) With M=1.5kg, m=3.1kg, I=10 Ns select values for k and L_0 to ensure $a_{\max} < 275g$ and $x>0$



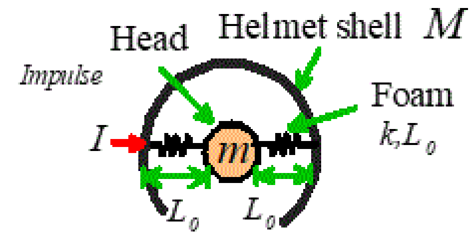
Approach

(1) Springs are undeformed during impact & so exerts no force on helmet ⇒ use impulse-momentum to get v_0

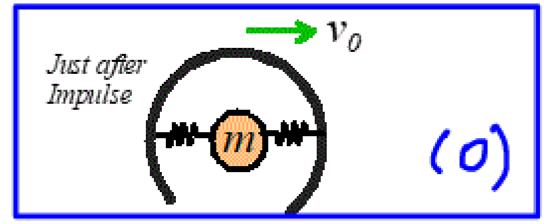
(2) No ext forces on helmet & head after impact
Conservative system ⇒ energy & momentum conserved

(3) Helmet & head have same speed @ max compression

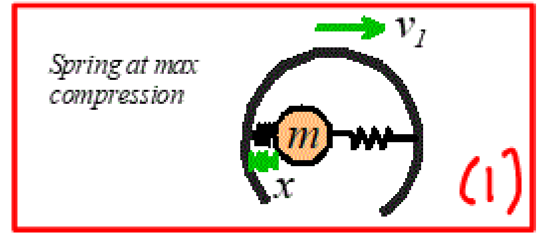
(a) Impulse - Momentum for helmet



$$\underline{I} = \underline{M}\,\underline{v_0} \Rightarrow \boxed{v_0 = I / M} \quad (1)$$



(b) Energy $\Rightarrow$ $T_1 + U_1 = T_0 + U_0$



$$\Rightarrow \frac{1}{2}(M+m)v_1^2 + 2\times\frac{1}{2}k(L_0-x)^2 = \frac{1}{2}M v_0^2 \quad (2)$$

Momentum $\Rightarrow$ $p_1^{TOT} = p_0^{TOT}$

$$\Rightarrow (M+m)\,v_1 = M\,v_0 \quad (3)$$

Subs (1) into (2) & (3) $\Rightarrow$ $\dfrac{I^2}{2(M+m)} + k(L_0-x)^2 = \dfrac{I^2}{2M}$

$$\Rightarrow \boxed{x = L_0 - I\sqrt{\frac{m}{2k(M+m)M}}}$$

(c) $\underline{F} = m\underline{a}$ for head $\Rightarrow m a_{max} = 2k(L_0 - x)$

$$\Rightarrow m a_{max} = I\sqrt{\frac{2km}{(\underline{M}+m)\underline{M}}} \Rightarrow \boxed{a_{max} = I\sqrt{\frac{2k}{(\underline{M}+m)\underline{M}\,m}}}$$

(d) $a_{max} < 275g \Rightarrow k \leq \frac{1}{2}(\underline{M}+m)\,\underline{M}\,m\left(\frac{a_{max}}{I}\right)^2$

$$\Rightarrow k < 778 \text{ kN/m}$$

$x > 0$ to avoid head hitting shell

$$\Rightarrow L_0 > I\sqrt{\frac{m}{2k(\underline{M}+m)\underline{M}}}$$

$$\Rightarrow \boxed{L_0 > 0.54 \text{ cm}}$$